A Simple Way of Calculating Cosmological Relic Density

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Abstract

A simple procedure is presented which leads to a dramatic simplification in
the calculation of the relic density of stable particles in the Universe.

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Any stable species χ contributes to the total mass-energy density in the Universe. If its number density cannot be reduced efficiently enough before it decouples from the thermal equilibrium, its relic abundance $\Omega_\chi h_0^2$ can be sizeable and it can affect the evolution and the age of the Universe. A conservative estimate that the Universe is at least 10 billion years old requires $\Omega_\chi h_0^2 < \Omega_{\mathrm{TOT}} h_0^2 < 1$ [1]. Furthermore, there is growing evidence for dark matter at both galactic and larger length scales [1] which would most likely require the existence of some type of exotic neutral particles. Since such particles are often present in many theories beyond the Standard Model, it is important to develop a simple practical procedure for calculating their relic abundance with enough precision.

Two groups [2, 3] have developed equivalent frameworks for properly calculating the relic density of χ's, including relativistic corrections. The method of Ref. [2] is in practice applicable away from poles and new final-state thresholds, which is most often the case. In Ref. [3] also the vicinity of poles and thresholds has been carefully studied. Essentially, one needs to calculate the thermally averaged product of the $\chi\bar{\chi}$ annihilation cross section and their relative velocity $\langle \sigma v_{\mathrm{rel}} \rangle$. In practice, one expands $\langle \sigma v_{\mathrm{rel}} \rangle = a + bx + \mathcal{O}(x^2)$ in powers of $x \equiv T/m_\chi = \mathcal{O}(1/20)$ in order to avoid difficult numerical integrations and approximates $\langle \sigma v_{\mathrm{rel}} \rangle$ by a and b. Both techniques give equivalent results [3, 4] in the overlapping region (away from poles and thresholds).

Unfortunately, in practice the actual calculation of even the first two terms of the expansion is typically very complicated and tedious. In this Letter I report on a dramatic technical simplification in practical applications of the method of Ref. [2].

Consider an annihilation of particles χ, $\bar{\chi}$ into a two-body final state. Furthermore, in many cases of interest, the final state particles have equal mass m_F. (A general case of unequal final state masses will be presented elsewhere [5].) Let the momenta of the two initial states χ, $\bar{\chi}$, and the two final states F, $\bar{F}$, be p_1, p_2 and k_1, k_2, respectively. Srednicki, *et al.*, introduce the function $w(s)$ defined as

$$w(s) \equiv \frac{1}{4} \int dLISP \, |\overline{\mathcal{M}}|^2 = E_1 E_2 \, \sigma v_{\mathrm{rel}}, \tag{1}$$

where $d\,LISP$ in this case takes the form

$$d\,LISP = (2\pi)^2 \delta^4(p_1 + p_2 - k_1 - k_2) \frac{d^3 k_1}{(2\pi)^3 2E_1} \frac{d^3 k_2}{(2\pi)^3 2E_2}, \tag{2}$$

where $E_{1,2} = \vec{k}_{1,2}^2 + m_F^2$, and $|\overline{\mathcal{M}}|^2$ is the square of the reduced matrix element for the annihilation process $\bar{\chi}\chi \to \bar{F}F$ summed over the spins of the final-state particles and averaged over the spins of the initial particles [2].

Denote

$$f \equiv |\overline{\mathcal{M}}|^2. \tag{3}$$

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In general $f = f(\vec{p}_1, \vec{p}_2, \vec{k}_1, \vec{k}_2)$. The integral Eq. (1) can be conveniently evaluated in the centre-of-mass frame (CM) in which $\vec{p}_2 = -\vec{p}_1$ and $\vec{k}_2 = -\vec{k}_1$. After a few elementary steps one obtains

$$w(s) = \frac{1}{2^6\pi}\sqrt{1 - \frac{4m_F^2}{s}} \int_{-1}^{1} d\cos\theta\, f(\vec{k}_2 = -\vec{k}_1, |\vec{k}_1| = \sqrt{s/4 - m_F^2}). \qquad (4)$$

Using the kinematic relation between the Mandelstam variables

$$t = (p_1 - k_1)^2 = m_\chi^2 + m_F^2 - \frac{s}{2}\left[1 - \sqrt{1 - \frac{4m_\chi^2}{s}}\sqrt{1 - \frac{4m_F^2}{s}}\cos\theta\right], \qquad (5)$$

one can express Eq. (4) as

$$w(s) = \frac{1}{2^5\pi}\frac{1}{s\sqrt{1 - 4m_\chi^2/s}} \int_{t_-(s)}^{t_+(s)} dt\, f(s,t), \qquad (6)$$

where $t_\pm \equiv t(\cos\theta = \pm 1) = t_0 \pm \Delta t$,

$$t_0(s) = m_\chi^2 + m_F^2 - \frac{s}{2} \qquad (7)$$

and

$$\Delta t(s) = \frac{s}{2}\sqrt{1 - \frac{4m_\chi^2}{s}}\sqrt{1 - \frac{4m_F^2}{s}} \qquad (8)$$

Notice that one can always express f as a function of the Mandelstam variables s and t only [2]. In particular, u can be eliminated by using the relation $s + t + u = 2m_\chi^2 + 2m_F^2$. Furthermore, in calculating the relic density it is convenient to introduce [2]

$$z \equiv \frac{s}{4m_\chi^2} \qquad (9)$$

in terms of which Eq. (6) can be simply rewritten as

$$w(z) = \frac{1}{2^7\pi m_\chi^2}\frac{1}{z\sqrt{1-z}} \int_{t_-(z)}^{t_+(z)} dt\, f(z,t). \qquad (10)$$

In order to calculate the relic abundance of χ's one needs to solve the Boltzmann (rate) equation. The actual quantity that appears in the rate equation is the thermally averaged product $\langle \sigma v_{\rm rel}\rangle$, which is usually approximated by $a + bx$, as mentioned above. One of the main results of Srednicki, *et al.* [2], was to show that

$$a = \frac{1}{m_\chi^2}w(z = 1) \qquad (11)$$

$$b = \frac{1}{m_\chi^2}\left[\frac{3}{2}\frac{dw(z = 1)}{dz} - 3w(z = 1)\right]. \qquad (12)$$

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While these formulae look deceptively simple, the actual calculations can be, and in practice *are*, very cumbersome and often virtually unmanageable. For example, in a relatively simple case of calculating the interference term between two t-channel amplitudes (with the masses of the exchanged particles denoted by μ_1 and μ_2) one needs to evaluate several integrals of the type $\int_{t_-}^{t_+} dt\, t^n (t - \mu_1^2)^{-1} (t - \mu_2^2)^{-1}$, and in general in calculating $w(z)$ one has to compute a whole multitude of more complicated integrals. Once such a (lengthy) expression for $w(z)$ is found, one needs to next take a derivative dw/dz which in general leads to even more cumbersome formulae. Additional highly non-trivial computational complications arise when the masses of the final state particles are not equal.

Below I show that a great deal of these difficulties can be avoided. In fact, one can avoid performing any integrals *completely*. First, one can always conveniently express [2] a and b in terms of "reduced" variables a^0 and b^0

$$a = \sqrt{1 - m_F^2/m_\chi^2}\, a^0, \tag{13}$$

$$b = \sqrt{1 - m_F^2/m_\chi^2} \left\{ \left[-3 + \frac{3}{4} \frac{m_F^2/m_\chi^2}{(1 - m_F^2/m_\chi^2)} \right] a^0 + b^0 \right\}. \tag{14}$$

I will show below that a^0 and b^0 can be written in a simple and elegant form as

$$a^0 = \frac{1}{2^5 \pi m_\chi^2} f(z = 1), \tag{15}$$

$$b^0 = \frac{1}{2^5 \pi m_\chi^2} \left[-3 m_\chi^2 \frac{\partial f(z=1)}{\partial t} + m_\chi^2 (m_\chi^2 - m_F^2) \frac{\partial^2 f(z=1)}{\partial t^2} + \frac{3}{2} \frac{\partial f(z=1)}{\partial z} \right] \tag{16}$$

and $f(z = 1)$ should be understood as $f(z = 1, t_0(z = 1))$, *etc.* Equations (15) and (16) are the main result of this Letter. They can also be readily generalized to the case of unequal final-state masses [5]. Higher order terms of the expansion can also be easily derived. It is clear that the whole procedure of calculating a^0 and b^0 has now been reduced to merely writing down $|\overline{\mathcal{M}}|^2$, substituting all the variables in $|\overline{\mathcal{M}}|^2$ in terms of z and t and next taking a few relatively simple derivatives. In fact, one can easily do all these steps entirely with the help of any advanced algebraic program. The truly difficult part of computing a^0 and b^0 - performing complicated integrations in deriving $w(z)$ - has been completely eliminated.

In order to prove Eqs. (15) and (16) notice that for any regular function $g(z, t(z))$ and its integrand $G(z) = \int dt\, g(z, t(z))$ one can show that

$$\lim_{z \to 1} \left[\frac{1}{\sqrt{1 - z}} G(z)|_{t_-(z)}^{t_+(z)} \right] = \lim_{z \to 1} \left[\frac{1}{\sqrt{1 - z}} \int_{t_-(z)}^{t_+(z)} dt\, g(z, t(z)) \right]$$

$$= 4 m_\chi^2 \sqrt{1 - m_F^2/m_\chi^2}\, g(z = 1, t_0(z = 1)) \tag{17}$$

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where I have used $\lim_{z\to 1} t_\pm = t_0(z=1)$, $\lim_{z\to 1}\Delta t = 2m_\chi^2\sqrt{1-m_F^2/m_\chi^2}\sqrt{1-z}$ and

$$\lim_{z\to 1} G(z,t_\pm) = \lim_{z\to 1}\Big[\quad G(z,t_0)\pm g(z,t_0)\Delta t + \frac{1}{2}\frac{\partial g(z,t_0)}{\partial t}(\Delta t)^2 \tag{18}$$
$$\pm\frac{1}{6}\frac{\partial^2 g(z,t_0)}{\partial t^2}(\Delta t)^3 + \mathcal{O}((\Delta t)^4)\Big].$$

(In deriving Eq. (16) one needs to apply the formula (17) twice (for f and $\partial f/\partial z$), and keep terms up to $(\Delta t)^3$ only in the expansion of f in order to cancel the term $(z-1)^{-3/2}$ which appears in dw/dz in Eq. (10).) Applying these formulae to Eqs. (10), (13), and (14) one obtains Eqs. (15) and (16) after a few simple steps.

As an example, consider the process $\chi\chi \to ZZ$ in the Minimal Supersymmetric Standard Model (MSSM) with χ being the lightest neutralino. The two s-channel diagrams include the exchange of the scalars h and H, while the four neutralinos χ_i^0 ($i=1,4$) (including $\chi \equiv \chi_1^0$) are exchanged in the t- and u-channels. Since neutralinos are Majorana particles, Eq. (3) now reads

$$f = \frac{1}{4}\sum_{helicity}\sum_{spin}|\mathcal{M}(\chi\chi\to ZZ)|^2 = \frac{1}{4}\sum_{helicity}\sum_{spin}|\mathcal{M}_s - \mathcal{M}_t + \mathcal{M}_u|^2 \tag{19}$$
$$= f^{(ss)} - f^{(st)} + f^{(su)} + f^{(tt)} - f^{(tu)} + f^{(uu)}.$$

After expressing the external momenta in terms of t, u, and z one finds, $e.g.$,

$$f^{(tt)}(z,t) = \frac{1}{8}\sum_{helicity}\sum_{spin}|\mathcal{M}_t|^2$$
$$= \frac{g^4}{\cos^4\theta_w}\sum_{i,j=1}^{4}(O_{1i}''^L)^2(O_{1j}''^L)^2\frac{1}{(t-m_{\chi_i^0}^2)(t-m_{\chi_j^0}^2)}\sum_{k=0}^{4}\sum_{l=0}^{1}f_{kl}^{(tt)}t^k z^l, \tag{20}$$
$$f^{(tu)}(z,t) = 2\Re\Big(\frac{1}{8}\sum_{helicity}\sum_{spin}|\mathcal{M}_t^\dagger\mathcal{M}_u|\Big)$$
$$= \frac{g^4}{\cos^4\theta_w}\sum_{i,j=1}^{4}(O_{1i}''^L)^2(O_{1j}''^L)^2\frac{1}{(t-m_{\chi_i^0}^2)(u-m_{\chi_j^0}^2)}\sum_{k=0}^{4}\sum_{l=0}^{2}f_{kl}^{(tt)}t^k z^l. \tag{21}$$

It is also clear that $f^{(uu)}(z,u) = f^{(tt)}(z,t)$. In the convention used here $O_{ij}''^L = -\frac{1}{2}(N_{i3}N_{j3} - N_{i4}N_{j4})$ (in the basis $(\widetilde{B}^0, \widetilde{W}_3^0, \widetilde{H}_b, \widetilde{H}_t)$), the matrix N_{ij} is real and the neutralino masses can be either positive or negative. The expressions for the coefficients $f_{kl}^{(tt)}$ and $f_{kl}^{(tu)}$, introduced above, are rather lengthy and will be given elsewhere [5]. I will neglect here also the contribution from the s-channel exchange [5].

Finally, one obtains

$$a^0 = \frac{g^4(m_\chi^2 - m_Z^2)}{4\pi\cos^4\theta_w}\sum_{i,j=1}^{4}\frac{(O_{1i}''^L)^2(O_{1j}''^L)^2}{\Delta_{\chi_i^0}\Delta_{\chi_j^0}} \tag{22}$$

and

$$
\begin{aligned}
b^0 \;=\;& \frac{g^4}{64\pi \cos^4\theta_{\rm w} m_Z^4} \sum_{i,j=1}^{4} \frac{(O_{1i}''^{L})^2 (O_{1j}''^{L})^2}{\Delta_{\chi_i^0}^3 \Delta_{\chi_j^0}^3} \times \\
& \Big[\Delta_{\chi_i^0}^2 \Delta_{\chi_j^0}^2 \big(48 m_{\chi_i^0} m_{\chi_j^0} m_\chi^4 + 32 m_{\chi_i^0} m_\chi^5 + 32 m_{\chi_j^0} m_\chi^5 + 32 m_\chi^6 \\
& - 48 m_{\chi_i^0} m_{\chi_j^0} m_\chi^2 m_Z^2 - 16 m_{\chi_i^0} m_\chi^3 m_Z^2 - 64 m_{\chi_j^0} m_\chi^3 m_Z^2 + 32 m_\chi^4 m_Z^2 \\
& + 36 m_{\chi_i^0} m_{\chi_j^0} m_Z^4 + 11 m_{\chi_i^0} m_\chi m_Z^4 + 29 m_{\chi_j^0} m_\chi m_Z^4 + 12 m_\chi^2 m_Z^4 + 32 m_Z^6 \big) \\
& + 2 \Delta_{\chi_i^0}^2 \Delta_{\chi_j^0} \big(4 m_{\chi_i^0} m_{\chi_j^0} m_\chi^6 - 24 m_{\chi_i^0} m_{\chi_j^0} m_\chi^4 m_Z^2 + 10 m_{\chi_i^0} m_\chi^5 m_Z^2 \\
& - 22 m_{\chi_j^0} m_\chi^5 m_Z^2 + 20 m_\chi^6 m_Z^2 + 20 m_{\chi_i^0} m_{\chi_j^0} m_\chi^2 m_Z^4 + 2 m_{\chi_i^0} m_\chi^3 m_Z^4 \\
& + 46 m_{\chi_j^0} m_\chi^3 m_Z^4 + 5 m_\chi^4 m_Z^4 - 12 m_{\chi_i^0} m_\chi m_Z^6 - 24 m_{\chi_j^0} m_\chi m_Z^6 - 25 m_\chi^2 m_Z^6 \big) \\
& + 2 \Delta_{\chi_i^0} \Delta_{\chi_j^0}^2 \big(28 m_{\chi_i^0} m_{\chi_j^0} m_\chi^6 - 24 m_{\chi_i^0} m_{\chi_j^0} m_\chi^4 m_Z^2 - 58 m_{\chi_i^0} m_\chi^5 m_Z^2 \\
& - 26 m_{\chi_j^0} m_\chi^5 m_Z^2 + 44 m_\chi^6 m_Z^2 - 4 m_{\chi_i^0} m_{\chi_j^0} m_\chi^2 m_Z^4 + 82 m_{\chi_i^0} m_\chi^3 m_Z^4 \\
& + 38 m_{\chi_j^0} m_\chi^3 m_Z^4 - 13 m_\chi^4 m_Z^4 - 24 m_{\chi_i^0} m_\chi m_Z^6 - 12 m_{\chi_j^0} m_\chi m_Z^6 - 31 m_\chi^2 m_Z^6 \big) \\
& + 16 \Delta_{\chi_i^0} \Delta_{\chi_j^0} \big(4 m_{\chi_i^0} m_{\chi_j^0} m_\chi^8 - 4 m_{\chi_i^0} m_{\chi_j^0} m_\chi^6 m_Z^2 - 6 m_{\chi_i^0} m_\chi^7 m_Z^2 \\
& - 6 m_{\chi_j^0} m_\chi^7 m_Z^2 + 4 m_\chi^8 m_Z^2 - 4 m_{\chi_i^0} m_{\chi_j^0} m_\chi^4 m_Z^4 + 12 m_{\chi_i^0} m_\chi^5 m_Z^4 \\
& + 12 m_{\chi_j^0} m_\chi^5 m_Z^4 - 5 m_\chi^6 m_Z^4 + 4 m_{\chi_i^0} m_{\chi_j^0} m_\chi^2 m_Z^6 - 6 m_{\chi_i^0} m_\chi^3 m_Z^6 \\
& - 6 m_{\chi_j^0} m_\chi^3 m_Z^6 - 2 m_\chi^4 m_Z^6 + 3 m_\chi^2 m_Z^8 \big) \\
& + 32 (\Delta_{\chi_i^0}^2 + \Delta_{\chi_j^0}^2) \big(m_\chi^6 m_Z^4 - 2 m_\chi^4 m_Z^6 + m_\chi^2 m_Z^8 \big) \Big]
\end{aligned}
\tag{23}
$$

where $\Delta_{i,j} \equiv m_Z^2 - m_\chi^2 - m_{\chi_{i,j}^0}^2$.

These expressions reduce nicely to the Eqs. (3.19.a) and (3.19.b) of Ref. [6] in the limit in which χ becomes an almost pure anti-symmetric higgsino $\widetilde{H}_A \equiv 1/\sqrt{2}\,(0,0,-1,1)$ (corresponding in the MSSM to $M_2 \gg \mu, m_Z$ [6]). In this limit the second lightest neutralino χ_2^0 is an almost pure symmetric higgsino $\widetilde{H}_S \equiv 1/\sqrt{2}\,(0,0,1,1)$ and it is almost mass-degenerate with χ, $|m_{\chi,\chi_2^0}| \simeq \mu(1 \mp \epsilon)$ and $m_{\chi_2^0} \simeq -m_\chi$. In Ref. [6] the contribution from the two heavier neutralinos, which in this limit are almost pure gauginos, has been neglected. However, as has been pointed out by Drees and Nojiri [7], they have in this limit non-negligible higgsino components which also contribute to the considered process. The expressions (22) and (23) are free from this problem because they are valid for any type of neutralino and include contributions from all four exchanged neutralinos.

The method presented here has also been tested on a number of other cases (*e.g.*, $\chi\chi \to (Z,h,H,A) \to \bar{f}f$), for which simple analytic expressions are available. A complete set of relevant expressions for the MSSM case will be presented elsewhere [5].

Note Added: After this work had been completed and submitted to a journal, I was made aware of Ref. [8] in which the expansion coefficients for $\sigma v_{\rm rel}$ were derived using a different method, without specifying a thermal averaging procedure. When these expressions are applied to the method of thermal averaging of Ref. [1], which does not include relativistic corrections [2, 3], they become different from the expression presented here by a factor $b - b_x = -\frac{3}{2}a$ [4].

Acknowledgments

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